February 23, 1854.

The Rev. BADEN POWELL, V.P., in the Chair.

The following communications were read:—

I. A paper entitled, "Continuation of the subject of a paper read Dec. 22, 1853, the supplement to which was read Jan. 12, 1854, by Sir FREDERICK POLLOCK, &c.; with a proof of Fermat's first and second Theorems of the Polygonal Numbers, viz. that every odd number is composed of four square numbers or less, and of three triangular numbers or less." By Sir FREDERICK POLLOCK, M.A., F.R.S. &c. Received February 23, 1854.

The object of this paper is in the first instance to prove the truth of a theorem stated in the supplement to a former paper, viz. "that every odd number can be divided into four squares (zero being considered an even square) the algebraic sum of whose roots (in some form or other) will equal 1, 3, 5, 7, &c. up to the greatest possible sum of the roots." The paper also contains a proof, that if every odd number $2n+1$ can be divided into four square numbers, the algebraic sum of whose roots is equal to 1, then any number n is composed of not exceeding three triangular numbers.

The general statement of the method of proof may be made thus: two theorems are introduced which connect every odd number with the gradation series, 1, 3, 7, 13, &c., of which the general term is $n+n^2+1$ or $4p^2\pm 2p+1$ (that is, the double of a triangular number $+1$), each term of which series can be resolved into four squares, the algebraic sum of the roots of which, $p, p, p, p+1$, or $p-1, p, p, p$ may manifestly be $=1$. By these theorems it is shown that every odd number is divisible into four squares, having roots capable of

forming as the sum of the roots 1, 3, 5, 7, &c. up to the greatest possible sum of the roots.

As the four square numbers which compose an odd number must obviously be three of them even and one odd, or three odd and one even, the differences of the roots among themselves must be the first odd and the third even, or *vice versâ*; and therefore these roots must have the sum of the first and third differences an odd number; the middle difference may be either odd or even.

The first of the theorems referred to, called by the author "Theorem P," is in substance this:—

Let r, s, t, v be the roots the squares of which compose any odd number N, such that $r+s+t+v=1$, and let each of these roots be increased by m; then $r+m$, $s+m$, $t+m$, $v+m$ will be the roots of the odd number $N+2m(2m+1)$; and $m-r$, $m-s$, $m-t$, $m-v$ the roots of the odd number $N+2m(2m-1)$; the sum of the roots in the first case being $4m+1$, and in the second $4m-1$. So that giving to m successively the values 0, 1, 2, 3, &c. in the general form $N+2m(2m\pm 1)$, a series will be formed in which the sums of the roots will be 1, 3, 5, 7, 9, &c., and the sums of their squares N, $N+2.1.1$, $N+2.1.3$, $N+2.2.3$, $N+2.2.5$, $N+2.3.5$, $N+2.3.7$, $N+2.4.7$, &c.; or N, $N+1.2$, $N+2.3$, $N+3.4$, $N+4.5$, $N+5.6$, $N+6.7$, $N+7.8$, &c. So that if p be the distance of any odd number in this series from N, the number will be $N+p(p+1)$, and the sum of its roots will be $2p+1$.

The conclusions to be drawn from this theorem are then stated:—

1. The greatest sum of the roots of the squares into which any odd number can be divided may be obtained: for let $2n+1$ be any odd number, and $2p+1$ the odd number to which the algebraic sum of its roots is required to be equal; then if p is such that $p(p+1)$ is less than $2n+1$, the number $2n+1$ can be resolved into squares the sum of whose roots is $2p+1$; otherwise it cannot.

2. The form of the roots of $2n+1$ may be found of which the algebraic sum is any possible odd number $2p+1$ except 1, provided all the odd numbers less than $2n+1$ possess the property of having the algebraic sum of their roots $=1$. For if from $2n+1$, $p(p+1)$ be taken, there will remain an odd number (N in Theorem P) such that, according to the condition stated, the algebraic sum of its roots $=1$; and in the series of roots and odd numbers formed from

these roots according to theorem P, p terms from N will be found the number $2n+1$ composed of squares the algebraic sum of whose roots is $2p+1$.

It thus appears that any odd number $2n+1$ can be divided into squares the sum of whose roots will equal 3, 5, 7, &c. (any possible odd number except 1) if the odd numbers below it can be divided into squares the sum of whose roots $=1$; and if it can be shown that its roots in some form will equal 1, then the theorem M will be true for that number and for every number below it.

This is illustrated by an example, and then another theorem, called "Theorem Q," is stated. In this a series of roots and odd numbers is formed by making the 1st and 3rd differences of the roots constant, but reversed every alternate term, and increasing or diminishing the middle difference by 1 each term;—or the middle difference is made constant and the 1st and 3rd vary. The sums of the roots thus oecome constant in every term of the series, but the sums of the squares of the roots increase, as in theorem P, by the even numbers 2, 4, 6, 8, &c., so that the increase at any number of terms p is $p(p+1)$, or the double of a triangular number.

By the application of these theorems to a variety of examples, it is shown how any odd number may be composed of four squares, such that the algebraic sum of their roots may equal 1.

The theorems P and Q, it is considered, connect every odd number with every other odd number, so as to make it impossible if one odd number be composed of four squares, but that every other odd number should likewise be so. It is pointed out in what manner every possible combination of numbers which can furnish the differences of the roots of any squares, not exceeding four, which can make an odd number, and the sum of which roots $=1$, can be derived from the gradation series, that is from $4p^2 \pm 2p+1$. The combined effect of the theorems P and Q is therefore to prove that every odd number must be composed of not exceeding four square numbers.

The author goes on to show that every number is composed of not exceeding three triangular numbers, by proving that if every odd number $2n+1$ can be divided into four square numbers the sum of whose roots $=1$, then n will be composed of not exceeding three triangular numbers. This is done by taking the differences of the

roots of $2n+1$, the algebraic sum of which roots is one, and diminishing the middle difference by theorem Q until it reaches a number nearest to half the sum of the first and third differences. The difference between $2n+1$ and the number thus obtained will be the double of a triangular number $=2T$. By the next step, the extreme differences are reduced until they are of the form m, $m+1$; and the difference between $2n+1-2T$ and the number thus obtained will again be the double of a triangular number $=2T^1$. The differences last obtained give the double of a triangular number $+1=2T''+1$. So that we find $2n+1=2T+2T'+2T''+1$. Consequently $n=$ the sum of three triangular numbers, if all the three operations be necessary; if not, to two or one triangular number only.

II. The first part of a paper "On a Class of Differential Equations, including those which occur in Dynamical Problems." By W. F. Donkin, M.A., F.R.S., F.R.A.S., Savilian Professor of Astronomy in the University of Oxford.

This paper is intended to contain a discussion of some properties of a class of simultaneous differential equations of the first order, including as a particular case the form (which again includes the dynamical equations),

$$x'_i=\frac{dZ}{dy_i},\quad y'_i=-\frac{dZ}{dx_i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

where $x_1 \ldots x_n$, $y_1 \ldots y_n$ are two sets of n variables each, and accents denote total differentiation with respect to the independent variable t; Z being any function of x_1 &c., y_1 &c., which may also contain t explicitly. The part now laid before the Society is limited to the consideration of the above form.

After deducing from known properties of functional determinants a general theorem to be used afterwards, the author establishes the following propositions.

If $x_1 \ldots x_n$ be n variables connected with n other variables $y_1 \ldots y_n$ by n equations of the form $y_i=\dfrac{dX}{dx_i}$ (X being a given function of $x_1 \ldots x_n$); then the equations obtained by solving these algebrai-